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Model of the Constricted Unit Cell Type for Isotropic Granular Porous Media

The constricted unit cell model for isotropic granular porous media developed by Payatakes, Tien, and Turian (1973) is extended here to take in account the random orientation of the flow channels. In the proposed model, each unit cell corresponds to a pore (cavern) and has two coaxial constricted inlet and outlet ports (throats). The unit cells have random dimensions and orientations, the distributions of which can be determined from simple experimental measurements. The flow through a unit cell is assumed to be identical to that through a segment of the corresponding periodically constricted tube. The model is applied to the case of creeping Newtonian flow. The solution to the flow problem is obtained by a collocation method. Permeabilities of typical packings are calculated without use of any adjustable parameters (such as tortuosities, etc.) and are found in excellent agreement with experimental values.

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SCOPE

There are many important processes taking place in porous media for which the converging-diverging character of the porous space is of primary importance. This has been recognized for some time (Petersen, 1958); Houpeurt, 1959), but only recently have major advances been made in this direction. Payatakes, Tien, and Turian (1973*a*, *b*) proposed a model for granular porous media involving unit cells of the constricted tube type and solved the associated problem of Newtonian flow through periodically constricted tubes with a novel finite-difference method of the stream function-vorticity type, retaining the inertial terms of the equation of motion. Payatakes, Tien, and Turian (1974*a*, *b*) and Payatakes, Brown, and Tien (1977) used the above porous media model to model successfully deep-bed filtration of liquid-solid suspensions. It was shown that the converging-diverging character of the porous space is essential in explaining and analyzing the transient behavior of depth filtration. Sheffield and Metzner (1976) published an incisive analysis of flow of nonlinear fluids through packed beds and concluded that the experimentally observed behavior can be explained by taking into account the converging-diverging character of the flow. Recent fundamental studies in the mobilization,

motion, and entrapment of oil ganglia during tertiary recovery by chemical flooding come to the common conclusion that the converging-diverging character of the porous space of oil bearing rocks is quintessential in analyzing this complex process (Melrose and Brandner, 1974; Slattery, 1974; Oh and Slattery, 1976; Stegermeir, 1976).

The main objective of the present study is the extension and improvement of the model proposed by Payatakes et al. (1973*a*, *b*) by taking into account the random orientation of the channels. This constitutes a fundamental and significant improvement, as numerous phenomena of transport in porous media involve substantial lateral contributions. Another marked improvement is the attainment of analytical expressions for the stream function and velocity components in the case of Newtonian creeping flow, using a collocation method.

CONCLUSIONS AND SIGNIFICANCE

A model for isotropic packed beds of monosized or nearly monosized grains is developed which incorporates several important features of the void space. These include the convergent-divergent character of the flow channels, the random dimensions and orientation of pores and constrictions, the curvature of the surface of the graphs, and the interference effect of neighboring grains. Furthermore, the number of pores per unit volume and the dimensions of the pores and constrictions of the model correspond reasonably closely to those of the real porous medium. This model is obtained through a generalization of the Payatakes et al. (1973*a*, *b*) model. In addition to determining the geometry, size distribution, and orientation angle distribution for the new porous media model, a collocation solution of creeping Newtonian flow through the unit cells was developed and is given in a companion communication (Neira and Payatakes, 1977). This is a highly desirable feature, since it gives the flow field in analytical form, rather than in the matrix form obtained in Payatakes et al. (1973*b*) with a finite difference scheme. Permeability values for typical packings are obtained analytically without the use of adjustable empirical parameters (tortuosity, etc.) and are found in excellent agreement with experimental data.

The intended applications of the model developed here are in the study of phenomena taking place in granular media that are influenced strongly by the geometry of the porous structure, such as axial and radial dispersion, axial and radial convective heat or mass transfer, non-Newtonian flow through granular media, deep-bed filtration, drainage, imbibition, etc. A modified version of the present porous media model taking into account the anisotropy and other special features of sandstones will be used for tertiary oil recovery studies. This work will appear in a future publication.

The complexity of most porous structures makes impossible the exact description of the solid boundaries, let alone the solution of the corresponding flow problem. Thus, an exact mathematical analysis of transfer processes in porous media is usually unattainable. In order to circumvent this difficulty and to attain a degree of quantitative understanding, idealizations of the porous space (models) must be adopted.

The objective of a porous media model is to provide a reasonable idealization of the geometrical structure of a particular class of porous media, based on which a transport process of interest can be treated mathematically. To this end, the model should incorporate the most relevant characteristics of the real porous medium, while its complexity should be held at a manageable level. From these considerations, it becomes clear that the porous media model which is most appropriate for a given modeling application depends not only on the porous medium, but also on the process under study. As a rule, a given porous media model fails when applied to the study of processes more complex than or radically different from those for which it was formulated.

A systematic study of models of porous media would be voluminous, and it is certainly beyond the scope of the present work; the interested reader is referred to the works by Scheidegger (1974), Bear (1972), Payatakes et al. (1973*a*), and van Brakel (1975).

There is substantial evidence (Payatakes et al., 1974*a*, *b*; Rajagopalan and Tien, 1976; Sheffield and Metzner, 1976; Payatakes et al., 1977) that the geometry assigned to the elementary flow channels by the Payatakes et al. (1973*a*) model is realistic and forms an adequate basis for the study of axial momentum transfer as well as of the complex transport processes of deep filtration through packed beds.

On the other hand, the Payatakes et al. (1973*a*) model cannot be used to describe transport phenomena in a plane which is normal to the main flow direction. The cause of this limitation lies in the assumption of perfect mixing during the conceptual transition from one unit bed element to the next, together with the rigid interconnection assigned to the unit cells. Numerous phenomena of transport in porous media involve significant lateral contributions, as, for instance, radial dispersion, radial heat and mass convection, secondary and tertiary oil recovery by displacement, etc. For such phenomena, the Payatakes et al. (1973*a*) model cannot be used in its present form.

A realistic model of granular porous media, which pro-

vides a basis for theoretical studies of axial as well as lateral transport phenomena, is formulated here.

FORMULATION OF THE MODEL

Basic Definitions

For the sake of clarity, certain definitions of idealized elements of the void space of the granular bed will be stated first. The curvature of the spheroidal surfaces of the grains gives rise to alternate enlargements and constrictions of the intergrain openings. The large cavities among the grains will be referred to as pores and the narrow channels connecting such cavities as constrictions (or throats). The constrictions immediately adjacent to a given pore will be called satellite constrictions (or throats). By elemental void space (EVS), we will denote the void space formed by a pore together with the neighboring volume limited by the planes which pass through the narrowest part of the satellite constrictions of the pore. Certainly, the shape, dimensions, and orientation of the EVS's are random variables.

Each EVS fulfills two functions. First, it can be considered as a conduit through which fluid flows in an overall direction at the expense of a small pressure drop. Second, it serves as a vessel in which a number of streams enter through an equal number of satellite throats and get mixed. The resulting mixed fluid stream leaves the EVS being distributed through the rest of the satellite throats.

As in the case of the model developed by Payatakes et al. (1973*a*), it will be assumed that constrictions and pores can be characterized by single characteristic lengths. Each constriction is characterized by the diameter of its most narrow cross section (common boundary of two adjacent EVS's). Let this diameter be denoted by d_c and its distribution function referred to as the constriction size distribution (CSD). Each pore is characterized by its effective pore diameter (diameter of sphere of equivalent volume) denoted by d_p. The distribution function of d_p will be referred to as the pore size distribution (PSD).

Graton and Fraser (1935) have shown the complicated geometry of the EVS units for some systematic packings of spheres. A quantitative description of the actual void space of a random packing is not practical or perhaps feasible. It is, however, possible to formulate a geometrically well-defined volume element, which resembles a given EVS in structure and also approximates its overall effect to flow. Such an idealized volume element will be called a unit cell (UC), as opposed to the EVS which is the corresponding real volume element. Then, one can consider the granular porous medium as an ensemble of UC's of appropriately distributed dimensions and orientation.

Conversion of EVS to UC

In successive steps of idealization, we will convert the EVS's into corresponding unit cells with geometry that makes them representative void elements of the packed bed and also permits a quantitative analysis of transport phenomena. These steps are shown in Figure 1 using a simplified two-dimensional representation of some typical pores and their adjoining (satellite) constrictions. The first abstraction will be made by taking into account the divergent-convergent character of the satellite constrictions. Using the description of the constrictions advanced in the P-T-T model, it will be assumed that each constriction resembles a piece of constricted tube. As can be seen in Figure 1, the pore volume can be considered as formed by the overlapping extensions of diverging tubes.

Despite the simplification already made, the structure of the EVS at this stage remains quite complex. The number of contiguous constrictions, their sizes, and their

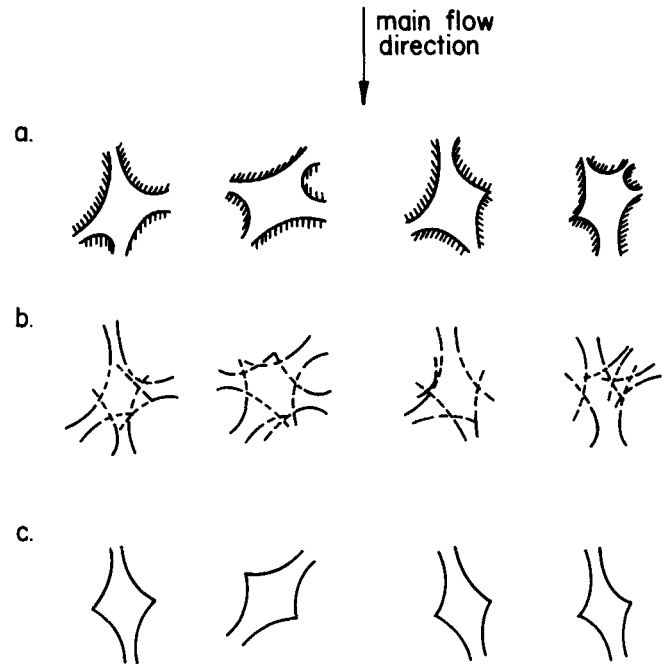

Fig. 1. Model formulation (two-dimensional depiction): a) prototype EVS, b) pore formed by constricted tubes, c) corresponding UC.

relative orientations are unknown random variables. In order to reduce this geometrical complexity to a manageable level, we continue the process of simplification as follows. Consider fluid flowing through a typical EVS, the structure of which may consist of any arrangement of interconnected constrictions, as the EVS's shown in Figure 1 (two-dimensional depiction). One of the main characteristics of flow through packed beds is the flow tendency to align its direction to a certain extent with the main flow direction. This tendency produces the characteristic zigzag path of fluid through a succession of what we have named EVS units. Thus, we can say that the major contribution to the flow is given by constrictions having an orientation close to the main flow (or pressure gradient) direction, and that the effect of constrictions close to the plane perpendicular to the main flow direction is expected to be relatively small. Based on this argument, it will be assumed that the net contribution of an EVS to flow can be approximated with that of an imaginary EVS having only two appropriately oriented constrictions. Consequently, it is both expedient and reasonable to approximate an EVS with a unit cell formed by two oppositely oriented constricted tubes, the common axis of which is parallel to the satellite constriction whose axial direction is the one nearest to the main flow direction. For further clarification, let us consider an EVS with constrictions having cubic arrangement as shown in Figure 2. Since we are considering an isotropic medium, all possible orientations of this EVS relative to the main flow direction are equally likely. For reasons already discussed, to each possible orientation of the EVS there will be a corresponding preferred orientation for the equivalent UC. The orientations of the ensemble of UC's representing the porous medium will be randomly distributed around the main flow direction. However, the angle between UC and main flow direction cannot be uniformly distributed in the range from 0 to $\pi/2$, as it might appear at the outset reasonable to assume. It can be seen easily that preferred UC orientations nearly perpendicular to the main flow direction are extremely improbable. For the example of Figure 2, those orientations are simply not possible. In general, we can say that owing to the high degree of interconnectivity of the pore structure, each EVS contributes significantly to

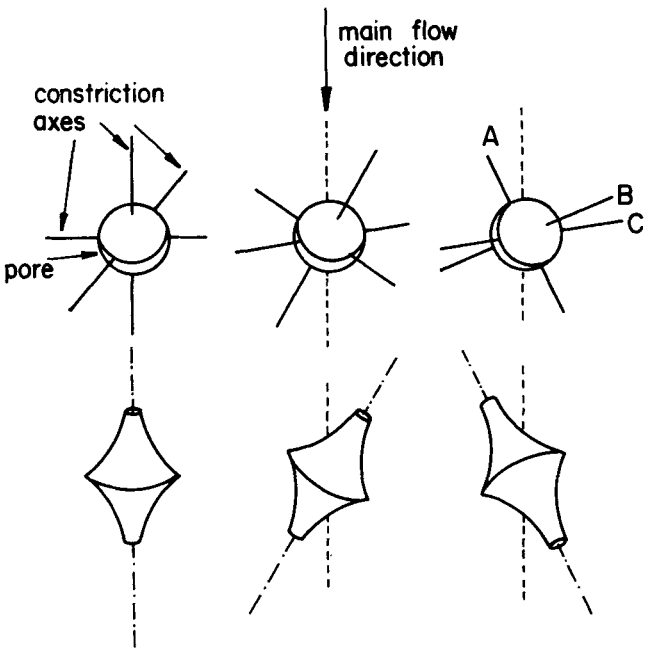

Fig. 2. Model formulation. Examples of preferred direction in the case of constricted tubes forming a cubic lattice.

the flow in the macroscopic transport direction. Accordingly, there will be a maximum possible angle between the axis of the corresponding UC and the main flow direction. Let that limiting angle be denoted by α.

In order to establish a lower bound for α, we will assume that the EVS's composing the void space of the medium have a cubic configuration, Figure 2. This geometry gives a number of pore-to-pore interconnections which exceeds the average in a randomly packed bed of grains. Hence, the value of α corresponding to this structure will be a lower bound to that for a packed bed of grains. Indeed, fewer constrictions per pore and irregular geometry are expected to give higher values of α. As assumed, the preferred unit cell orientation corresponding to each EVS is given by the satellite constriction direction which is closest to the main flow direction. Therefore, the largest angle which is possible for the preferred direction will occur when the main flow follows a direction which is equidistant from three adjacent constrictions, say, A, B, and C in Figure 2. That angle will represent the limiting angle α. For the case of cubic arrangement of the satellite constrictions, it will be denoted by α_c. A simple geometric consideration gives $\alpha_c = \tan^{-1}\sqrt{2} \approx 0.3041\pi(54.74 \text{ deg})$. This value should be considered as a lower bound of α. In packed beds of grains, the average number of satellite throats is less than six (about five). This and the random nature of the packing lead to an expected value of α larger than α_c. On the other hand, α is certainly smaller than 90 deg, the absolute limit. Thus, α should lie in the range $54.74 < \alpha < 90$ deg. As a first approximation, the value of α can be fixed at the mid value of that interval, that is, $\alpha \approx 0.402\pi(72.37 \text{ deg})$. [It should be noted that applying similar considerations to a regular tetrahedric structure as assumed by Josselin de Jong (1958), the value obtained for α is $\cos^{-1}(1/3) \approx 0.3918\pi$ (70.53 deg)].

Calculated friction factor values through use of the present model with the above value of α are in remarkable agreement with experimental data.

Orientation of a Unit Cell (UC)

Consider a UC of the porous medium, Figure 3. Its orientation in space is given by θ and ϕ, where θ is the

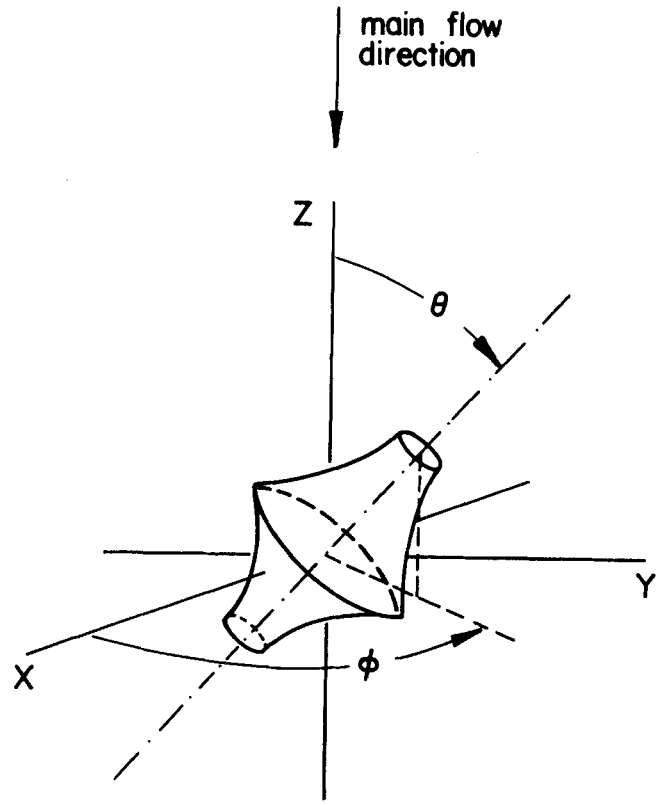

Fig. 3. Typical unit cell. The values of angles θ and ϕ determine its orientation.

angle between the UC axis and the positive Z axis (main direction of the flow) and ϕ is the angle between the X axis and the projection of the UC axis on the XY plane, measured counterclockwise. It will be assumed that the orientation angles of the UC's, θ, and ϕ in the granular bed are uniformly distributed in the ranges $[0, \alpha]$ and $[0, 2\pi]$, respectively, where α is the limiting angle already estimated. Therefore, the probability that a UC has an orientation with angles in the ranges θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$ is given by

$$f_{\theta\phi}d\theta d\phi = \frac{\sin\theta d\theta d\phi}{A} \quad \text{for} \quad \begin{cases} 0 \leq \theta \leq \alpha \\ 0 \leq \phi \leq 2\pi \end{cases}$$

where $f_{\theta\phi}$ is the distribution function of the UC orientation, and A is the corresponding normalization constant. It is shown readily that $A = 2\pi(1 - \cos\alpha)$; hence

$$f_{\theta\phi} = \frac{\sin\theta}{2\pi(1 - \cos\alpha)} \quad \text{for} \quad \begin{cases} 0 \leq \theta \leq \alpha \\ 0 \leq \phi \leq 2\pi \end{cases} \qquad (1)$$

Unit Cell Dimensions

The next step in the development of the model is the determination of the dimensions of the unit cells and their distribution. These should agree as closely as possible with the corresponding dimensions of the EVS's of the actual porous medium. As already mentioned, the dimensions given to the unit cells in the P-T-T model satisfy this objective fairly well. For this reason, we will adopt their results.

To this end the following major assumptions are made: the wall of a UC is defined by two symmetric parabolae of revolution, and all the UC's that compose the granular medium are geometrically similar, though not necessarily equal. The last assumption implies that the geometry of an

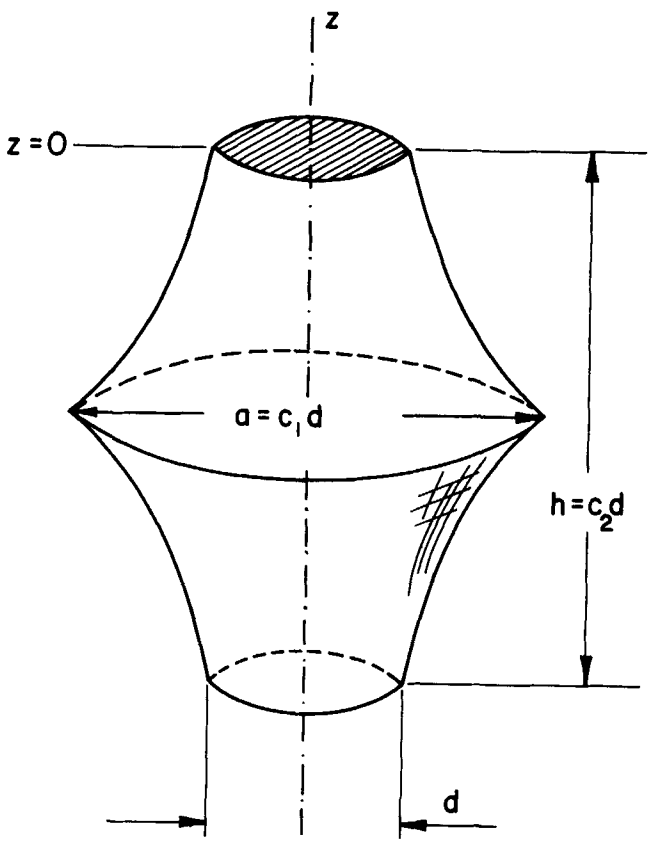

Fig. 4. Dimensions of a UC. d, a, and h are random variables; c_1 and c_2 are constants.

UC can be prescribed by a single dimension. Considering a typical UC, as shown in Figure 4, we will set as in Payatakes et al. (1973*a*)

$$a = c_1 d \tag{2}$$

and

$$h = c_2 d \tag{3}$$

The constriction diameter d is an independent random variable, whereas h and a are, according to Equations (2) and (3), dependent random variables. It is argued in Payatakes et al. (1973*a*) that the assumed proportionality between those variables is, to a certain extent, found in real beds of randomly packed grains. In general, in such beds large constrictions are found among large and/or loosely packed grains; hence, large constrictions usually connect large pores. An analogous argument can be made for small constrictions. It is seen, therefore, that in a porous medium composed of uniform or nearly uniform grains, the characteristic dimensions are expected to be strongly correlated, rather than independent of each other, due mainly to the considerable degree of geometric regularity imposed by the presence of the grains.*

The assumptions expressed by Equations (2) and (3) are not justified in the case of consolidated porous media. In the latter case, the correlation between d, a, and h is expected to be weak, and it could be assumed that they are independent random variables.

Consider an EVS with effective diameter (diameter of sphere of equal volume) d_p. Let, also, d_c be the effective diameter of that satellite constriction of the EVS which is most closely aligned with the main flow direction. Consider now the UC corresponding to the EVS under consideration. We will set

$$d = d_c \tag{4}$$

and

$$a = d_p \tag{5}$$

The size distribution of d_c (or d) is the constriction size distribution (CSD), as defined in the above. According to these assumptions, it follows that the geometry of UC's of a given granular bed is completely known if the CSD, the geometric constants c_1 and c_2, and the number of UC's per unit bed volume are determined.

Constriction Size Distribution (CSD) and Constants c_1 and c_2

The CSD function f_{d_c} will be estimated from the saturation vs. capillary pressure curve which is obtained from an initial drainage experiment. Drainage and imbibition or penetration curves have been used extensively in the determination of characteristic lengths of the void space of porous media, (usually capillary pore size distributions).

A recent review of these methods is given by van Brakel (1975). In the present study, we will adopt the analysis advanced by Payatakes et al. (1973*a*). The results of that analysis are summarized below:

$$d = d_c \text{ (random variable)} \tag{4}$$

$$a = c_1 d_c = \left[\frac{\epsilon(1 - S_{wi})}{(1 - \epsilon)} \frac{<d_g^3>}{<d_c^3>} \right]^{1/3} d_c \tag{6}$$

$$h = c_2 d_c = \frac{<d_g>}{<d_c>} d_c \tag{7}$$

where S_{wi} is the irreducible saturation value, obtained from the initial drainage experiment, ϵ is the macroscopic porosity of the bed, and where the first and third moments of d_c and d_g are calculated from the constriction size distribution and the grain size distribution, respectively. The constriction diameter frequency function is to be determined from the initial drainage curve using

$$f_{d_c} = \frac{p_s^3}{\int_{S_{wi}}^1 p_s^3 dS} \frac{dS}{dd_c} = \frac{p_s^5}{4\gamma_{12} \int_{S_{wi}}^1 p_s^3 dS} \left(-\frac{dS}{dp_s} \right) \tag{8}$$

In deriving Equation (8), it was assumed that the constriction diameter is given by (as a matter of fact, defined by)

$$d_c = \frac{4\gamma_{12} \cos\theta_c}{p_s} \tag{9}$$

θ_c is assumed to vanish ($\theta_c \approx 0$, $\cos\theta_c \approx 1$). This is a valid approximation for initial water drainage (receding menisci) in a water wetted medium. Under different experimental conditions, the contact angle usually depends on the particular liquid-solid system as well as on the rate and direction of the motion of the interface (advancing or receding menisci). The dependence of the contact angle of wetting fluids on the rate of interface motion in the case of imbibition (advancing menisci) is larger than that in the case of drainage (receding menisci). This is the main reason for choosing the initial water drainage curve for the

* Dullien (1975) suggested that this assumption is not supported by his analysis of data composed of mercury porosimetry curves and photomicrography curves. Such data are indeed valuable in the study of porous structures. On the other hand, they do not contain by themselves sufficient information concerning the degree of correlation between the value of d at a given constriction and the values of a and h at the adjacent pores. No analysis of a porosimetry curve and a photomicrography curve should attempt to draw conclusions that require information that is not contained in recoverable form in the data, especially when such an analysis is itself based on several assumptions. A substantially more direct experiment must be devised for this particular measurement. Until a satisfactory experimental technique of this type is developed, it seems acceptable to use Equations (2) and (3); they are reasonable as well as expedient, leading to geometrically similar unit cells and a single flow problem.

determination of the CSD over water imbibition curves. The cumulative constriction size distribution is obtained by integrating Equation (8):

$$F_{d_c}(d_c) = \int_{d_{c,\min}}^{d_c} f_{d_c}(d_c)\,dd_c = \frac{\int_{S_{wi}}^{S} p_s^3 dS}{\int_{S_{wi}}^{1} p_s^3 dS} \tag{10}$$

$F_{d_c}(d_c)$ is the number fraction of constrictions with size less than d_c.

The wall of each UC is assumed to be composed of two symmetric parabolae of revolution, Figure 4. The wall radius r_w as a function of the axial coordinate z is given by

$$r_w(z) = \begin{cases} \dfrac{d}{2} + 2(a-d)\left(\dfrac{z}{h}\right)^2 & \text{for } 0 \le z \le \dfrac{h}{2} \\ \dfrac{d}{2} + 2(a-d)\left(1-\dfrac{z}{h}\right)^2 & \text{for } \dfrac{h}{2} \le z \le h \end{cases} \tag{11}$$

Number of UC's per Unit Volume of Bed

Since we have assigned to each elemental void space (EVS) a unit cell (UC), the numbers of EVS's and UC's per unit volume of bed are equal. The determination of N_p will be made so that the macroscopic porosity of the actual granular bed is conserved. A total void balance over a unit volume of bed gives

$$\epsilon = N_p \langle V_{UC} \rangle \tag{12}$$

where $\langle V_{UC} \rangle$ is the mean volume of UC's composing the medium. The volume of a UC of characteristic diameter d is determined by

$$V_{UC}(d) = \int_o^h \pi r_w^2(z)\,dz \tag{13}$$

where $r_w(z)$ is given by Equation (11). Integration of Equation (13) and substitution of Equations (2), (3), and (4) give

$$V_{UC}(d_c) = \frac{\pi c_2}{60}(8 + 4c_1 + 3c_1^2) d_c^3 \tag{14}$$

Averaging and substituting this expression into Equation (12), one gets

$$N_p = \frac{60\epsilon}{\pi c_2(8 + 4c_1 + 3c_1^2)} \frac{1}{\langle d_c^3 \rangle} \tag{15}$$

where c_1 and c_2 are given by [see Equations (6) and (7)]

$$c_1 = \left[\frac{\epsilon(1 - S_{wi})}{(1-\epsilon)} \frac{\langle d_g^3 \rangle}{\langle d_c^3 \rangle} \right]^{1/3} \tag{16}$$

$$c_2 = \frac{\langle d_g \rangle}{\langle d_c \rangle} \tag{17}$$

Flow Through the Unit Cells

The flow through a unit cell is obtained exactly as in the case of the Payatakes et al. (1973*a*, *b*) model, namely, by postulating that it is identical to the flow through a segment of the corresponding periodically constricted tube. The latter problem was solved by a finite-difference method, retaining the inertial terms of the equation of motion, in Payatakes et al. (1973*b*). To solve the flow problem, one has to specify the pressure drop across a given unit cell. If ∇P is the macroscopic pressure gradient, the pressure drop along a given unit cell with length h and orientation angle θ is assumed to be

$$-\Delta P_{UC} = |\nabla P| h \cos\theta = c_2 |\nabla P| d \cos\theta \tag{18}$$

A collocation solution of the creeping flow of a Newtonian fluid through the dimensionless unit cell is given in a companion communication (Neira and Payatakes, 1977). If ψ^* is the dimensionless stream function, the collocation solution is given by

$$\psi^*(\xi,\eta) \approx \psi_N^*(\xi,\eta) = \psi_o^*(\xi) + \sum_{k=1}^{N} C_k \xi(1-\xi)^{i+1} \cos(j-1)2\pi\eta \tag{19}$$

where

$$\left.\begin{aligned} &\psi_o^*(\xi) = (r_1^{*2}/2)(1-\xi)^2 \\ &k = (j-1)n_r + i \\ &i = 1, \ldots, n_r; \quad j = 1, \ldots, n_z \\ &N = n_r n_z \end{aligned}\right\} \tag{20}$$

and C_k are the expansion coefficients. In the above, (ξ, η) is a new coordinate system defined by

$$\xi(r^*, z^*) = \left(\frac{r^*}{r_w^*}\right)^2 \left[\left[1 - \left(\frac{r^*}{r_w^*}\right)^2 \right] \gamma + 1 \right] \tag{21}$$

$$\eta(z^*) = z^* \tag{22}$$

with

$$\gamma(z^*) = w - 2(w+1)\left(\frac{r_w^*}{r_2^*}\right) + (w+3)\left(\frac{r_w^*}{r_2^*}\right)^2 \tag{23}$$

$$r^* = \frac{r}{h}, \quad z^* = \frac{z}{h}, \quad r_w^* = \frac{r_w}{h} \tag{24}$$

and w is an adjustable parameter ($w = 1$ is suggested for typical geometries in packed beds of grains).

Values of C_k for periodically constricted tube geometries corresponding to typical packings are given in Table 1 for $N = 9$ and 16 interior collocation points.

The velocity components can be obtained readily from

$$v_r^* = \frac{v_r}{v_o} = \frac{1}{r^*}\left[\left(\frac{\partial \xi}{\partial z^*}\right)_{r^*} \left(\frac{\partial \psi^*}{\partial \xi}\right)_{\eta} + \left(\frac{\partial \psi^*}{\partial \eta}\right)_{\xi} \right],$$

$$v_z^* = \frac{v_z}{v_o} = -\frac{1}{r^*}\left(\frac{\partial \xi}{\partial r^*}\right)_{z^*} \left(\frac{\partial \psi^*}{\partial \xi}\right)_{\eta} \tag{25}$$

where v_o is the average velocity at the most narrow cross section.

Under creeping flow conditions

$$\frac{\Delta P_{UC}}{\rho v_o^2} = \Delta P^*_{UC}(r_1^*, r_2^*) = \frac{\Delta P_1^*(r_1^*, r_2^*)}{N_{Re}} \tag{26}$$

where

$$r_1^* = \frac{1}{2}\frac{d_c}{h}, \quad r_2^* = \frac{1}{2}\frac{a}{h} \tag{27}$$

$$N_{Re} = \frac{h\rho v_o}{\mu} \tag{28}$$

and where ΔP_1^* is the dimensionless pressure drop along one tube segment at $N_{Re} = 1$, neglecting inertial effects. Values of ΔP_1^* for various values of r_1^* and r_2^* are given in Neira and Payatakes (1977) in chart form (see also Table 1). Equations (18) and (26) give

$$v_o(d_c, \theta) = \frac{c_2^2 |\nabla P|}{\mu(-\Delta P_1^*)} d_c^2 \cos\theta, \quad \text{any } \phi \tag{29}$$

The volumetric flow rate through a unit cell is given by

$$q(d_c,\theta) = \frac{\pi c_2^2 |\nabla P|}{4\mu(-\Delta P_1^*)} d_c^4 \cos\theta, \quad \text{any } \phi \tag{30}$$

The Z component of the linear momentum of the fluid in a unit cell is given by

$$m_Z(d_c, \theta) = \int_{V_{UC}} \rho[v_z(r, z; d_c, \theta)]_Z dV$$

where v_z is the velocity component parallel to the cell axis and $[v_z]_Z$ is its projection in the Z direction. The above equation can be expressed as

$$m_Z(d_c, \theta) = \int_0^{c_2 d_c} \int_0^{r_w(z)} \rho v_z(r, z; d_c, \theta) \cos\theta 2\pi r dr dz \tag{31}$$

We also have

$$q(d_c, \theta) = \int_0^{r_w(z)} v_z(r, z; d_c, \theta) 2\pi r dr \tag{32}$$

Hence

$$m_Z(d_c, \theta) = \rho c_2 d_c \cos\theta \, q(d_c, \theta) \tag{33}$$

which together with Equation (30) gives

$$m_Z(d_c, \theta) = \frac{\rho\pi c_2^3 |\nabla P|}{4\mu(-\Delta P_1^*)} d_c^5 \cos^2\theta \tag{34}$$

Flow Through the Bed

The permeability k, friction factor f_s, and the superficial Reynolds number $(N_{Re})_s$ are defined by

$$v_s = \frac{k}{\mu} |\nabla P| \tag{35}$$

$$f_s = \frac{\langle d_g \rangle}{2\rho v_s^2} |\nabla P| \tag{36}$$

and

$$(N_{Re})_s = \frac{\rho \langle d_g \rangle v_s}{\mu} \tag{37}$$

The following relationship between these three quantities holds:

$$f_s (N_{Re})_s = \frac{\langle d_g \rangle^2}{2k} \tag{38}$$

The Z component of the total linear momentum in a volume V_m of the medium is given by

$$M_Z = \rho\epsilon V_m \langle v_Z \rangle = \rho V_m v_s \tag{39}$$

where $\langle v_Z \rangle = v_s/\epsilon$ is the mean interstitial velocity in the Z direction. The total linear momentum can also be evaluated as the sum of the linear momentum contributions of all the UC's in the volume V_m:

$$M_Z = V_m N_p \int_E m_Z(d_c, \theta) dE \tag{40}$$

Here dE is the probability of occurrence of a unit cell with constriction diameter in the range d_c to $d_c + dd_c$ and orientation angles in the ranges θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$; E denotes the domain of the random variables d_c, θ, and ϕ. Since these variables are independent, the probability dE is given by the product of their individual occurrence probabilities. We have

$$dE(d_c, \theta, \phi) = \frac{\sin\theta}{2\pi(1 - \cos\alpha)} f_{d_c} dd_c d\theta d\phi \tag{41}$$

The domain E is given by $d_{c,\min} \leq d_c \leq d_{c,\max}$, $0 \leq \theta \leq \alpha$ and $0 \leq \phi \leq 2\pi$. As a first approximation, the limiting angle has been set equal to 0.402π (72.37°) (see above). Substitution of Equations (34) and (41) into Equation (40) and integration of the latter lead to

$$\frac{M_Z}{V_m} = \frac{N_p \rho\pi(1 + \cos\alpha + \cos^2\alpha) c_2^3 \langle d_c^5 \rangle |\nabla P|}{12\mu(-\Delta P_1^*)} \tag{42}$$

Equations (15), (39), and (42) together give

$$\langle v_Z \rangle = \frac{5(1 + \cos\alpha + \cos^2\alpha) c_2^2 \langle d_c^5 \rangle |\nabla P|}{\mu(8 + 4c_1 + 3c_1^2) \langle d_c^3 \rangle (-\Delta P_1^*)} \tag{43}$$

TABLE 1. COLLOCATION SOLUTION COEFFICIENTS $C_k \times 10^3$ AND DIMENSIONLESS PRESSURE DROP ΔP_1^* FOR TYPICAL UNIT CELL GEOMETRIES WITH $N = 9$ AND $16 (n_r = n_z)$

Sample*	Glass beads				Sand			
Porosity: ϵ	0.38		0.40		0.41		0.47	
r_1^*	0.1760		0.1885		0.1685		0.1820	
r_2^*	0.3975		0.4035		0.4020		0.4300	
	$N = 9$	$N = 16$	$N = 9$	$N = 16$	$N = 9$	$N = 16$	$N = 9$	$N = 16$
$k = 1$	−0.97875	−1.0494	−0.19664	−0.26547	−1.8548	−1.9283	−3.0828	−3.2230
2	1.4257	2.1602	1.0762	1.6875	1.9180	2.8480	2.9123	4.3449
3	−0.52929	−1.7628	−0.41883	−1.4372	−0.68650	−2.2442	−0.99823	−3.2862
4	1.8851	0.57124	0.82805	0.47316	3.3274	0.71362	5.2980	1.0245
5	−2.3059	1.9652	−1.6454	0.87622	−3.1893	3.4531	−4.7596	5.5758
6	0.90074	−3.6420	0.71120	−2.6715	1.1459	−4.9882	1.6433	−7.5511
7	0.96978	3.2212	1.3316	2.5820	0.56393	4.1106	0.18460	5.9706
8	1.0046	−1.0551	0.60654	−0.87504	1.4193	−1.3034	2.0161	−1.8571
9	−0.40809	0.99503	−0.34077	1.4647	−0.45795	0.43953	−0.62411	−0.16625
10	—	2.1563	—	1.3243	—	3.1637	—	4.7461
11	—	−2.3822	—	−1.8826	—	−2.9949	—	−4.2562
12	—	0.78527	—	0.66473	—	0.93176	—	1.2978
13	—	0.11234	—	−0.07948	—	0.38139	—	0.72777
14	—	−1.2166	—	−0.82886	—	−1.7056	—	−2.4393
15	—	1.1884	—	0.98854	—	1.4108	—	1.9191
16	—	−0.36233	—	−0.32574	—	−0.39976	—	−0.53130
$-\Delta P_1^*$	122.49	122.45	111.57	111.54	128.39	128.34	112.20	112.16

* Source of geometrical data Payatakes (1973). The geometries studied correspond to two randomly packed beds of 30 × 40 glass spheres with porosities 0.38 and 0.40 and to two beds of 20 × 30 sand with porosities 0.41 and 0.47. The dimensionless unit cell geometries (and the coefficients C_k) remain unchanged if the sizes of the entire population of grains or spheres are multiplied by an arbitrary factor, without changing the arrangement (and the porosity).

Table 2. Experimental and Theoretical Analysis of Typical Random Packings of Grains

Experimental	Packing		Theoretical	Packing	
Exp. quantities	Glass beads	Sand	Calc. quantities	Glass beads	Sand
	0.40	0.47	UC geometry		
Sieve analysis:			c_1	2.141	2.363
$<d_g>$, mm	0.470*	0.714†	c_2	2.652	2.747
$<d_g^3>$, mm^3	1.038×10^{-1}	3.640×10^{-1}	r_1^*	0.1885	0.1820
Initial drainage curve:			r_2^*	0.4035	0.4300
S_{wi}	0.111	0.127	N_p, mm^{-3}	15.15	4.46
$<d_c>$, mm	0.177	0.260	Collocation solution:		
$<d_c^3>$, mm^3	6.272×10^{-3}	2.143×10^{-2}	$-\Delta P_1^*$	111.54	112.16
$<d_c^5>$, mm^5		2.203×10^{-3}			
Flow measurement at low $(N_{Re})_s$:	2.523×10^{-4}		Predictions for small $(N_{Re})_s$:		
$f_s \cdot (N_{Re})_s$	473	406	$f_s \cdot (N_{Re})_s$	472	385
k, mm^2	2.33×10^{-4}	6.28×10^{-4}	k, mm^2	2.34×10^{-4}	6.63×10^{-4}

* The size distribution of glass beads was determined from magnified photographs of a population of over 500 spheres.
† Geometric mean of the nominal diameters of U.S. Standard No. 20 and No. 30 sieves.

Finally, Equations (35) and (43) give

$$k = \frac{5\epsilon(1 + \cos\alpha + \cos^2\alpha)c_2^2<d_c^5>}{(8 + 4c_1 + 3c_1^2)<d_c^3>} \frac{1}{(-\Delta P_1^*)} \quad (44)$$

Two terms have been separated on the right-hand side of the above equation. The first one is a geometric term which is only function of the structure of the porous space of the bed. The second one constitutes a hydrodynamic term which, again, depends exclusively on the geometry of the UC. A chart for the calculation of $\Delta P_1^*(r_1^*, r_2^*)$ is given in Neira and Payatakes (1977). Values of ΔP_1^* for geometries corresponding to typical packings are given in Table 1.

Comparison with Experimental Permeability Values

Permeabilities for two typical random packings are calculated with the present model and compared with the experimental values reported by Payatakes et al. (1973*a*) as a first step towards validation. One case corresponds to a bed composed of glass beads with nominal diameter of 470 μm and with porosity 0.40. The other corresponds to a bed composed of sand 20 × 30 grains and with porosity 0.47. The characteristics of those beds and the experimental results are given in Table 2. The data used to calculate the various moments of d_c were derived from the saturation vs. capillary pressure curves reported in Figure 4 of Payatakes et al. (1973*a*) and Equation (8).

The calculated values for the reported cases are also given in Table 2. The predicted friction factor in the case of the bed of glass beads differs from the experimental value by −0.2%. For the case of the bed of sand, that difference is −5.2%. It should be emphasized that the model predictions are based on theoretical considerations and independent measurements; no adjustable parameter (such as tortuosity factor, etc.) was used to fit the permeability data.

On the Interconnectivity of Unit Cells

In order to model certain phenomena (such as drainage, imbibition, motion of oil-ganglia through porous media, etc.), precise information as to how the unit cells are interconnected to each other is required.

The unit cells of the present model can be assumed as being interconnected with lines that have no volume and present no resistance to flow. Such a possibility is shown in Figure 5, where four unit cells are connected to the same mixing-distributing point, two contributing fluid to it and two receiving fluid from it. Owing to the assumed random orientation of the unit cells, the resulting network is irregular. This is a serious drawback as it would render bookkeeping in drainage, imbibition, or immiscible displacement calculations problematic. It appears, then, that for such applications a regular network (say, tetrahedric) would be preferable. In the latter case, the feature of random orientation of unit cells is abandoned, although lateral transport contributions are retained. This approach is used in a forthcoming work pertaining to immiscible oil displacement.

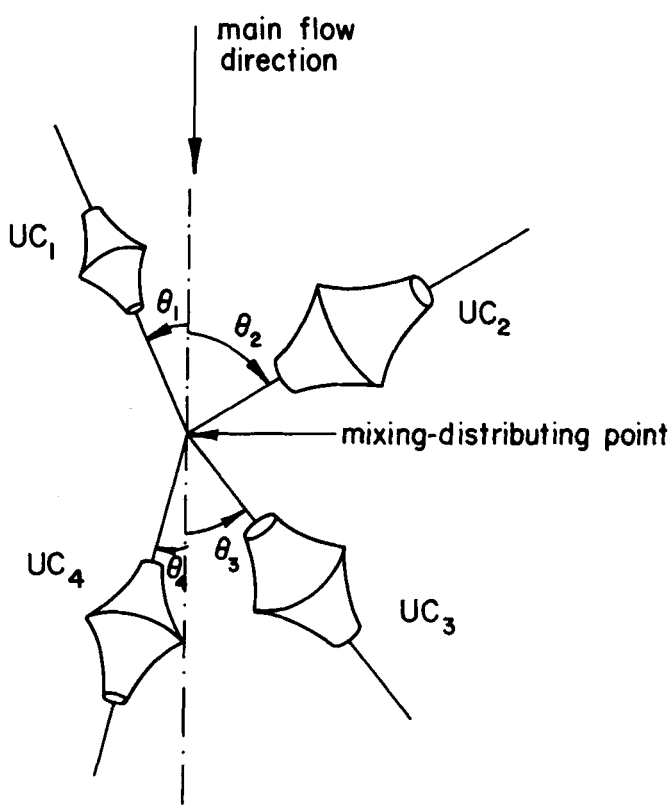

Fig. 5. Elemental network unit. The sizes and orientations of the unit cells are independent random variables.

NOTATION

a = maximum diameter of a unit cell
C_k = expansion coefficients, Equation (19)
c_1, c_2 = constants, given by Equations (16) and (17), respectively
d = minimum diameter of a unit cell
d_c = constriction diameter, defined by Equation (9)
d_g = grain diameter
d_p = pore (cavern) diameter
F_{dc} = cumulative constriction diameter distribution
f_{dc} = constriction diameter frequency function
f_s = friction factor of packed bed, Equation (36)
$f_{\theta\phi}$ = distribution function of the unit cell orientation, Equation (1)
h = unit cell length
k = permeability
M_Z = Z component of total linear momentum in the bed
m_Z = Z component of linear momentum in a unit cell
N = number of collocation points
N_p = number of pores (caverns) per unit volume
N_{Re} = Reynolds number, Equation (28)
$(N_{Re})_s$ = superficial Reynolds number, Equation (37)
n_r, n_z = number of collocation points in the radial and axial direction, respectively
p_s = suction applied in the drainage experiment
q = flow rate through a unit cell
r = radial coordinate in a unit cell
$r_w(z)$ = unit cell wall radius
r^* = dimensionless radial coordinate in a unit cell, Equation (24)
r_w^* = dimensionless unit cell wall radius, Equation (24)
r_1^*, r_2^* = minimum and maximum dimensionless radii of a unit cell, Equation (27)
S = corresponding to p_s saturation value
S_{wi} = irreducible saturation value
V_m = volume of porous medium
V_{UC} = unit cell volume
v_s = superficial velocity through the bed
v_r, v_z = radial and axial velocity components in a unit cell
v_o = mean velocity at the constriction of a unit cell
v_r^*, v_z^* = dimensionless radial and axial velocity components in a unit cell, Equation (25)
$\langle v_Z \rangle$ = mean interstitial velocity
$\langle v_z \rangle_Z$ = component of v_z in the Z direction (main flow direction)
w = parameter, Equation (23)
Z = coordinate parallel to the main flow direction, Figure 3
z = axial coordinate in a unit cell
z^* = dimensionless axial coordinate in a unit cell, Equation (24)

Greek Letters

α = upper bound of angle θ
α_c = value of α for cubic arrangement of the constrictions
γ = function defined by Equation (23)
γ_{12} = water-air interfacial tension
ΔP_{UC} = pressure drop along a unit cell
ΔP_{UC}^* = dimensionless pressure drop along a unit cell, Equation (26)
ΔP_1^* = value of ΔP_{UC}^* at $N_{Re} = 1$ (neglecting inertial effects)
ϵ = porosity
η = axial coordinate, Equation (22)
θ = angle between the axis of a unit cell and the main flow direction, Figure 3
θ_c = contact angle in the drainage experiment
μ = dynamic viscosity
ξ = new radial coordinate in a unit cell, Equation (21)
ρ = fluid density
ϕ = UC orientation angle measured counterclockwise in the plane normal to the main flow direction, Figure 3
ψ^* = dimensionless stream function
ψ_N^* = approximation to ψ^* from N collocation point solution, Equation (19)
ψ_o^* = initial guess in trial function, Equation (20)

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